

Crack propagation at the interface between viscoelastic and elastic materials

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Abstract

Crack propagation in viscoelastic materials has been understood with the use of Barenblatt cohesive models by many authors since the 1970's. In polymers and metal creep, it is customary to assume that the relaxed modulus is zero, so that we have typically a crack speed which depends on some power of the stress intensity factor. Generally, when there is a finite relaxed modulus, it has been shown that the toughness increases between a value at very low speeds at a threshold toughness G_0 , to a very fast fracture value at G_∞ , and that the enhancement factor in infinite systems (where the classical singular fracture mechanics field dominates) simply corresponds to the ratio of instantaneous to relaxed elastic moduli.

Here, we apply a cohesive model for the case of a bimaterial interface between an elastic and a viscoelastic material, assuming the crack remains at the interface, and neglect the details of bimaterial singularity. For the case of a Maxwell material at low speeds the crack propagates with a speed which depends only on viscosity, and the fourth power of the stress intensity factor, and not on the elastic moduli of either material. For the Schapery type of power law material with no relaxation modulus, there are more general results. For arbitrary viscoelastic materials with nonzero relaxed modulus, we show that the maximum toughness enhancement will be reduced with respect to that of a classical viscoelastic crack in homogeneous material.

Keywords:

1. Introduction

The problem of viscoelastic crack growth is of fundamental importance see the recent review of Rodriguez et al. (2020). Early investigations (Gent and Schultz, 1972, Barquins and Maugis 1981, Gent, 1996, Gent & Petrich 1969, Andrews & Kinloch, 1974, Barber *et al*, 1989, Greenwood & Johnson, 1981, Maugis & Barquins, 1980) noticed a steady state subcritical crack propagation with an enhanced work of adhesion G with respect to the adiabatic value G_0 namely

$$\frac{G}{G_0} = 1 + \left(\frac{V}{V_0}\right)^n \quad (1)$$

where V is velocity of peeling of the contact/crack line, a characteristic velocity was defined as $V_0 = (ka_T^n)^{-1}$ and k, n are (supposedly) constants of the material, with $0 < n < 1$ where a_T is the WLF factor to translate results at various temperatures T (Williams, Landel & Ferry, 1955).

From a more fundamental perspective, initially a paradox was identified by Graham (1969). Namely, since the stress field singularity is the classical inverse square root of elastic materials, at the crack tip we have infinite frequency and therefore an "elastic" material, which does not explain dissipation and speed dependence of toughness enhancement (Rice, 1978). But the paradox was solved by various authors (Schapery, 1975, Greenwood & Johnson, 1981, Barber *et al.*, 1989, Greenwood, 2004 and others, see a review by Bradley et al.1997) using cohesive Barenblatt or Dugdale models removing the singularity in a cohesive zone whose size increases with speed (because of the toughness enhancement). Another school explains enhancement with estimating the dissipation (de Gennes, 1996), by the "viscoelastic trumpet" crack model, as the crack shape is different in the inner "glassy region", the intermediate "liquid region", and finally in the outer soft "rubbery" region. On this second school, notable improvements have been made by Persson & Brener (2005) who clarified the relevant range of frequencies involved in the estimating dissipation, and gave a solution for a general viscoelastic material.

Both schools suggest results qualitatively of the form (1) and introduce the cohesive strength of the material σ_c and therefore introduce the length

scale

$$a_0 = \frac{G_0 E_0}{2\pi\sigma_c^2} \quad (2)$$

where E_0 is the relaxed modulus (the modulus at zero frequency) and σ_c is the cohesive stress.

Also, all these models suggest looking at the remote stress intensity factor K as applied in remote regions as giving an effective toughness $K_{Ic}^2(V) = G(V) E_0$, where E_0 is expected in remote regions, and hence obtain the maximum toughness enhancement as

$$\frac{G(\infty)}{G_0} = \frac{E_\infty}{E_0} \quad (3)$$

For many polymeric or rubbery materials this ratio is very large possibly an increase of 3 or 4 orders of magnitude. Here, $G(\infty)$ stands for $G(V \rightarrow \infty)$ and E_∞ is instantaneous modulus or the modulus observed at infinite frequency of oscillatory loading. Remark however that in much of the literature on metal creep or polymers like that reviewed in Bradley et al.(1997), it is often assumed that the rheology corresponds to a zero relaxed modulus $E_0 = 0$ and hence there is no lower threshold for crack propagation. These studies obtain a power law for the crack propagation speed, of the type

$$V \propto K_I^m \quad (4)$$

where m depends on details of the rheology. Notice that the Gent-Schultz kind of result Eq. (1) is fully compatible with this scaling, when one considers $G \gg G_0$ so that (1) can be written as $V \propto K_I^{2/n}$ i.e. $m = 2/n$. For a standard material it can be shown that in the intermediate range of velocities $n = 1/2$ and hence $V \propto K_I^4$.

Therefore, as clearly explained by Wang et al (2016), the literature on ceramics or metal creep and on non cross-linked polymers see viscoelasticity as an "apparent weakening", since they compare to the elastic fast fracture limit, while the polymers literature shows viscoelastic results as a mechanism for enhanced dissipation "an apparent toughening" with respect to the threshold for the start of subcritical crack growth.

However, analysis of the case of a bimaterial crack between an elastic material and a viscoelastic one has not been attempted before to the best of our knowledge, at least not from a simplified perspective as we shall provide here. This is surprising, since the bimaterial crack problem has received

much attention for the elastic-elastic case (there are at least two papers with more than 1000 citations, for example, Suo & Hutchinson, 1990, and Comninou, 1977), while in many cases interface cracks occur between a rubbery or polymeric material peeling from an elastic surface (see Kendall's classical paper, Kendall (1975) again with nearly 1000 citations). This topic is now an emerging area of technology in adhesives and in Nature-inspired attachment systems, and in many cases the polymer will exhibit viscoelasticity. Therefore an analysis of the the problem is timely. Indeed, we show here a simple generalization of the cohesive model treatment for a bimaterial interface with viscoelasticity, obtaining some (approximate) closed form results.

2. A bimaterial crack propagation

If we consider a bimaterial interface with a semi-infinite crack, see Fig.1, where one material is elastic and the other viscoelastic, we can assume a cohesive model and write the Energy Release Rate (ERR) as

$$G = G_e + G_v = G_0 \quad (5)$$

where G_e is the ERR in the elastic material, and G_v the ERR in the viscoelastic material.

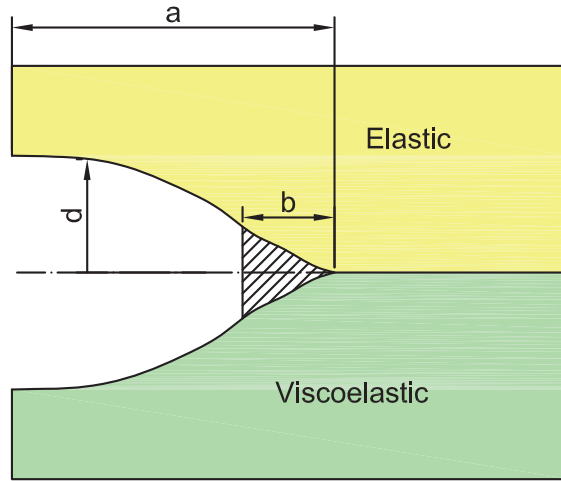


Fig.1 - The crack of size $a \gg b$ where b is a cohesive region, propagating at the interface between an elastic and a viscoelastic material

We consider the classical Schapery-Greenwood Dugdale cohesive model, and to address a semi-infinite crack, so we assume the conditions for K-field

dominance, or Small Scale Cohesion (SSC) to be satisfied. In this case we have simply

$$G_e = \frac{1}{2} \frac{K_I^2}{E} \quad ; \quad G_v = \frac{1}{2} K_I^2 C_{eff} \left(\frac{b}{V} \right) \quad (6)$$

where we consider plane stress for simplicity. For the viscoelastic material, we have introduced an effective compliance C_{eff} given by (Rice, 1978)

$$C_{eff} \left(\frac{b}{V} \right) = \int_0^1 C \left(\frac{b}{V} - \frac{b}{V} \lambda \right) \frac{df}{d\lambda} d\lambda \quad (7)$$

where $C(t)$ is the viscoelastic compliance, relating the instantaneous strain $\varepsilon(t) = \int_{-\infty}^t C(t-\tau) \frac{d\sigma(\tau)}{dt} d\tau$ to the history of stress $\sigma(t)$ in uniaxial conditions. The function $f(\lambda)$ is the opening (stretch) in the cohesive zone from the Dugdale model, given approximately in a bimaterial by

$$f(\lambda) = \sqrt{\lambda} - \frac{1}{2} (1 - \lambda) \ln \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \quad (8)$$

and the cohesive stress σ_c cancels the singularity (in the Dugdale form where the cohesive strength is constant and uniform in the cohesive region). Under SSC (small scale cohesion), the length of this cohesive zone is approximately

$$b = \frac{\pi}{8} \left(\frac{K_I}{\sigma_c} \right)^2 \quad (9)$$

In the appendices we justify the use of the Mode I stress intensity factor instead of the complex stress intensity factor in the rigorous treatment of bimaterial interfaces, and hence the approximations introduced in the last 3 equations.

3. A Maxwell material

We are not interested in giving a full solution for a general viscoelastic material, as we shall discuss qualitative features later. A convenient case is however that of a Maxwell material which has no relaxed modulus. In this case

$$C(t) = \frac{1}{E_\infty} + \frac{t}{\mu} \quad (10)$$

where μ is viscosity. A full simple analytical solution is possible then by performing the integration (7)

$$C_{eff} \left(\frac{b}{V} \right) = \frac{1}{E_\infty} + \frac{b}{3\mu V} \quad (11)$$

Hence, substituting in the energy balance equation (5) (6, 11) we get¹

$$G = \frac{1}{2} \frac{K_I^2}{E} + \frac{1}{2} K_I^2 C_{eff} \left(\frac{b}{V} \right) = \frac{1}{2} K_I^2 \left(\frac{1}{E} + \frac{1}{E_\infty} + \frac{b}{3\mu V} \right) = G_0 \quad (12)$$

and using (9) this reduces to

$$\frac{1}{2} K_I^2 \left(\frac{1}{E^*} + \frac{\pi}{24} \left(\frac{K_I}{\sigma_c} \right)^2 \frac{1}{\mu V} \right) = G_0 \quad (13)$$

where we define an combined modulus

$$\frac{1}{E_\infty^*} = \frac{1}{E} + \frac{1}{E_\infty} \quad (14)$$

This (13) leads then to the simple solution

$$\mu V = \frac{\pi}{24} \frac{K_I^4 E_\infty^* / \sigma_c^2}{2G_0 E_\infty^* - K_I^2} \quad (15)$$

Obviously, the critical condition of fast fracture is at $V \rightarrow \infty$ when (15) gives

$$K_{Ic,\infty}^2 = 2G_0 E_\infty^* \quad (16)$$

Vice versa, for very low $K_I^2 \ll 2G_0 E^*$, we can write from (15)

$$\mu V_{low} = \frac{\pi}{24} \frac{K_I^4}{2G_0 \sigma_c^2} \quad (17)$$

a simple scaling which does *not* depend on E^* , and so on neither of the elastic moduli of the elastic and viscoelastic materials (but just on the viscosity). This is a typical results, when the viscous fracture-length scale is small and the stress field has the classical K-field dominance, see eqt.25 of Wang et al (2016), where the scaling is common in small-scale damage-zone models of creep-rupture in linear materials like the model of Cocks and Ashby (1982).

¹Obviously we could add the elastic halfplane compliance in the $C(t)$ and C_{eff} function, and then remove the elastic contribution to G , and the results would not change.

4. The Schapery power law forms of creep-compliance

More general exponents in the power law could be found if one uses a more general rheology for the material (but still assuming $E_0 = 0$) i.e. with more than one relaxation times or a continuous spectrum of relaxation times, like in much of the literature on metal creep or polymers as that reviewed by Bradley et al.(1997). Indeed, Schapery (1975a,1975b) shows that one can use

$$C_{eff}(t) = \frac{1}{E_\infty} + C_1 \left(d \frac{b}{V} \right)^n \quad (18)$$

where $d \simeq 1/3$ is a corrective factor which depends very weakly on $n \in [0, 1]$ (in our previous Maxwell model we showed it should be equal exactly to $1/3$), and C_1 is a generalized viscosity with dimensions $[\frac{s^{-n}}{MPa}]$. For our bimaterial interface, repeating the analysis we obtain

$$G = \frac{1}{2} \frac{K_I^2}{E} + \frac{1}{2} K_I^2 C_{eff} \left(\frac{b}{V} \right) = \frac{1}{2} K_I^2 \left(\frac{1}{E} + \frac{1}{E_\infty} + C_1 \left(d \frac{b}{V} \right)^n \right) = G_0 \quad (19)$$

or

$$\frac{1}{2} K_I^2 \left(\frac{1}{E_\infty^*} + C_1 \left(\frac{\pi d}{8} \right)^n \left(\frac{K_I}{\sigma_c} \right)^{2n} \frac{1}{V^n} \right) = G_0 \quad (20)$$

leading to

$$V = \left(\frac{\pi d}{8} \right) \frac{(C_1 E_\infty^*)^{1/n}}{(2E_\infty^* G_0 - K_I^2)^{1/n}} \left(\frac{K_I^{2+2/n}}{\sigma_c^2} \right) \quad (21)$$

which at low load leads to a scaling with $K_I^{2+2/n}$

$$V_{low} = \left(\frac{\pi d}{8} \right) \frac{(C_1)^{1/n}}{(2G_0)^{1/n}} \left(\frac{K_I^{2+2/n}}{\sigma_c^2} \right) \quad (22)$$

The last two equations generalize Schapery's result to a crack on a bimaterial interface using his assumed generalized rheology. Notice that in the power law regime the role of elastic modulus disappears again for this class of materials. Notice also that fast fracture occurs at the same level of stress intensity factor, independently of n , as we expect.

We can rewrite (21) by taking the dimensionless factor $\widehat{K}^2 = \frac{K_I^2}{2E_\infty^* G_0}$, and defining $a_\infty^* = \frac{G_0 E_\infty^*}{2\pi\sigma_c^2}$

$$\widehat{V} = \frac{V}{(C_1 E_\infty^*)^{1/n} a_\infty^*} = \frac{\pi^2}{6} \frac{\widehat{K}^{2+2/n}}{(1 - \widehat{K}^2)^{1/n}} \quad (23)$$

Fig.2 shows some example plots of equation (23).

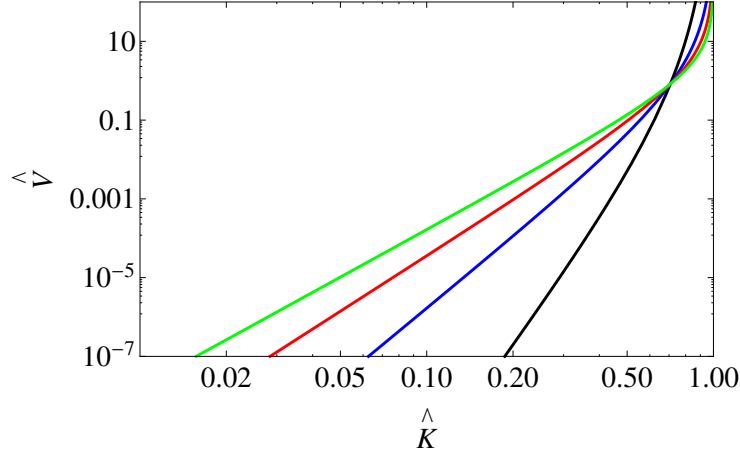


Fig.2 - The dimensionless speed of subcritical crack propagation from equation (23) for $n = 0.25, 0.5, 0.75, 1$ (black, blue, red, green solid lines) in a viscoelastic/elastic bimaterial interface for a semi-infinite crack, as a function of dimensionless stress intensity factor

5. The case of non-zero relaxed modulus

Our Maxwell material analytical result is clearly instructive, and the generalization to the Schapery creep compliance form has proved quite useful in the engineering literature, see Bradley et al.(1997), as applied to quite general form of polymers or metal creep (Cocks and Ashby, 1982). It is hard to give exact simple results for the most general rheology and is outside of the scope of the present investigation. One point of interest for any viscoelastic material with finite relaxed modulus $E_0 > 0$, is that obviously $C_{eff}(V \rightarrow 0) = \frac{1}{E_0}$, so that we would have the trivial energy balance equation

$$G = \frac{1}{2} \frac{K_I^2}{E} + \frac{1}{2} \frac{K_I^2}{E_0} = G_0 \quad (24)$$

and so we now define another combined modulus

$$\frac{1}{E_0^*} = \frac{1}{E} + \frac{1}{E_0}$$

to obtain

$$K_{Ic,0}^2 = 2G_0 E_0^* \quad (25)$$

Comparing this with (16), we obtain the maximum amplification

$$A_{\max} = \frac{K_{Ic,\infty}^2}{K_{Ic,0}^2} = \frac{E_{\infty}^*}{E_0^*} = \frac{E_{\infty}}{E_0} \frac{\left(\frac{E_0}{E} + 1\right)}{\left(\frac{E_{\infty}}{E} + 1\right)} \quad (26)$$

and since we expect that for polymers or rubbery materials $\frac{E_{\infty}}{E_0} \gg 1$ (which is why we get the toughness amplification)

$$A_{\max} = \frac{K_{Ic,\infty}^2}{K_{Ic,0}^2} < \frac{E_{\infty}}{E_0} \quad (27)$$

Thus the limiting toughness enhancement is less than that for the crack with the viscoelastic material on both faces. Fig.3 shows the maximum amplification A_{\max} from (26) (black thick solid line) as a function of the ratio E/E_0 , as compared with an even simpler approximation, $A_{\max} = \frac{E}{E_0}$, which shows the maximum amplification is of the same order of the ratio E/E_0 in an intermediate range.

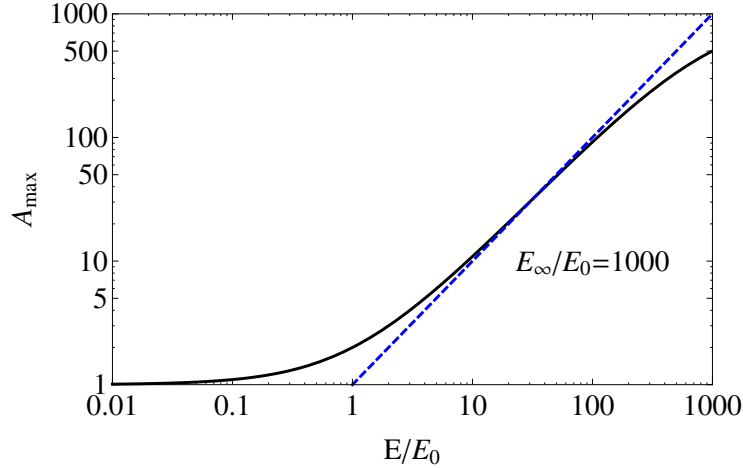


Fig.3 - The maximum toughness amplification A_{\max} from (26) (black thick solid line) as a function of the ratio E/E_0 , where E is the elastic modulus of the elastic material, and E_0 the relaxed modulus of the viscoelastic one.

Blue dashed line is the simplified form $A_{\max} = \frac{E}{E_0}$

6. Conclusions

We have obtained a simple closed form solution for the subcritical propagation of a crack at the interface between an elastic and a viscoelastic material, in the form of a Maxwell material, or for the more general creep compliance rheology of Schapery. We find that the "elastic" fracture occurs for $K_{Ic,\infty}^2 = 2G_0E_\infty^*$ where the "equivalent" modulus is the inverse of the sum of the compliances of the elastic material and the (instantaneous value) of the viscoelastic material. The subcritical crack propagation for the Maxwell material depends only on the viscosity of the material, and not on any of the elastic moduli, and scales at low speeds with the fourth order of the stress intensity factor. For the Schapery creep compliance form, the results are quite similar, except there is a more general power law dependence. We also argue that for any more general viscoelastic material constitutive equation, having a relaxed modulus $E_0 > 0$, the toughness amplification would be less than that expected for the viscoelastic semi-infinite crack in a homogeneous material.

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9. Appendix

The complex stress intensity factor due to a set of point forces applied to the surface of the crack at $x_1 = \tilde{x}_1$ is (Rice and Sih, 1965)

$$K = \sqrt{\frac{2}{\pi}} \cosh \pi \varepsilon \frac{Q + iR}{(l - \tilde{x}_1)^{\frac{1}{2} + i\varepsilon}} \quad (28)$$

where the x_1 -axis is the bimaterial interface, the crack occupies $x_1 \leq l$, where $l = b$ will be the cohesive zone size, and ε is the usual bimaterial index

$$\varepsilon = \frac{1}{2\pi} \ln \frac{1 - \beta}{1 + \beta} \quad (29)$$

where β is Dundurs' bimaterial constant, $i = \sqrt{-1}$, R represents a pair of horizontal, equal and opposite point forces and Q a pair of equal and opposite vertical forces. These forces are defined per unit thickness. The stress ahead of the crack on the x_1 -axis is (Rice, 1988)

$$\sigma_{22} + i\sigma_{12} = \frac{K}{\sqrt{2\pi}} (x_1 - l)^{-\frac{1}{2} + i\varepsilon} \quad (30)$$

The stresses in the cohesive zone are uniform and such that $\sigma_{22} + i\sigma_{12} = \sigma_c + i\sigma_s$, where σ_c is the cohesive stress and σ_s is a shear stress arising as a reaction to constraints on shearing motion relative to the interface. The net stress intensity factor is zero, so

$$\sqrt{\frac{2}{\pi}} \cosh \pi \varepsilon (\sigma_c + i\sigma_s) \int_0^b \frac{d\tilde{x}_1}{(b - \tilde{x}_1)^{\frac{1}{2} + i\varepsilon}} = K_A \quad (31)$$

where the cohesive zone occupies $0 \leq x_1 \leq b$, and K_A is the far-field applied stress intensity factor. This integrates to give

$$\sqrt{\frac{8}{\pi}} \cosh \pi \varepsilon (\sigma_c + i\sigma_s) \sqrt{bb^{-i\varepsilon}} = (1 - 2i\varepsilon) K_A \quad (32)$$

We represent the cohesive zone stress as

$$\sigma_c + i\sigma_s = \Sigma e^{i\varphi} \quad (33)$$

the applied stress intensity factor as

$$K_A = \sqrt{K_A \bar{K}_A} e^{i\psi} \quad (34)$$

and note that

$$b^{-i\varepsilon} = e^{-i\varepsilon \ln b} \quad (35)$$

and

$$1 - 2i\varepsilon = (1 + 4\varepsilon^2)^{\frac{1}{2}} e^{-i \arctan 2\varepsilon} \quad (36)$$

As a result, we find that

$$\varphi = \psi + \varepsilon \ln b - \arctan 2\varepsilon \quad (37)$$

$$b = \frac{\pi (1 + 4\varepsilon^2) K_A \bar{K}_A}{8 \cosh^2 \pi \varepsilon T^2} \quad (38)$$

and

$$T^2 = \frac{\sigma_c^2}{\cos^2(\psi + \varepsilon \ln b - \arctan 2\varepsilon)} \quad (39)$$

Note that Eq. (37,39) must be solved simultaneously, probably by iteration. However, for possible material combinations that exclude auxetics $\varepsilon \ll 1$ (Rice, 1988). In that case, a good, 1st order estimate of b is the result for homogeneous systems

$$b = \frac{\pi K_A^2}{8 \sigma_c^2} \quad (40)$$

where K_A is the mode I stress intensity factor in the far-field and we have assumed pure Mode I far-field loading.

Now, return to the point force solution Eq.(28) and add a second (auxiliary) set of point forces at $x_1 = \hat{x}_1$ (notice that usually $\hat{x}_1 \neq \tilde{x}_1$) so that the stress intensity factor is

$$K = \sqrt{\frac{2}{\pi}} \cosh \pi \varepsilon \left[\frac{F_2 + iF_1}{(l - \tilde{x}_1)^{\frac{1}{2} + i\varepsilon}} + \frac{P_2 + iP_1}{(l - \hat{x}_1)^{\frac{1}{2} + i\varepsilon}} \right] \quad (41)$$

The complex conjugate of K is

$$\bar{K} = \sqrt{\frac{2}{\pi}} \cosh \pi \varepsilon \left[\frac{F_2 - iF_1}{(l - \tilde{x}_1)^{\frac{1}{2} - i\varepsilon}} + \frac{P_2 - iP_1}{(l - \hat{x}_1)^{\frac{1}{2} - i\varepsilon}} \right] \quad (42)$$

The resulting energy release rate is (Rice, 1988)

$$G(F_1, F_2, P_1, P_2, l) = \frac{c_1 + c_2}{16 \cosh^2 \pi \varepsilon} K \bar{K} \quad (43)$$

where, in plane strain,

$$c_1 = \frac{8(1 - \nu_1^2)}{E_1} \quad (44)$$

and

$$c_2 = \frac{8(1 - \nu_2^2)}{E_2} \quad (45)$$

with E_i being Young's moduli and ν_i Poisson ratios. The subscript 1 indicates the material in $x_2 \geq 0$ and the 2 indicates the material in $x_2 \leq 0$.

Note that both sides of Eq. (43) are real. Let the displacements at $x_1 = \hat{x}_1$ on the top surface of the crack be $c_1(\delta_1, \delta_2)/2$, both components defined to be real. The displacements in the bottom surface of the crack are $c_2(\delta_1, \delta_2)/2$. Note that

$$\delta_1 = \delta_1(F_1, F_2, P_1, P_2, l) \quad (46)$$

and

$$\delta_2 = \delta_2(F_1, F_2, P_1, P_2, l) \quad (47)$$

Furthermore, using the following generalized Castigliano's theorem

$$\delta_1 = -\frac{\partial \Psi(F_1, F_2, P_1, P_2, l)}{\partial P_1} \quad (48)$$

$$\delta_2 = -\frac{\partial \Psi(F_1, F_2, P_1, P_2, l)}{\partial P_2} \quad (49)$$

and

$$G = -\frac{\partial \Psi(F_1, F_2, P_1, P_2, l)}{\partial l} \quad (50)$$

where Ψ is the total potential energy, sum of the strain energy and the potential energy of the applied loads. Notice that (48,49) reduce to the classical Castigliano's theorem for a linear system, in which $\Psi = -U$ where U is strain energy.

As a result, as noted by Burns et al. (1978) we have Maxwell relationships

$$\left(\frac{\partial \delta_1}{\partial l} \right)_{P_1} = \left(\frac{\partial G}{\partial P_1} \right)_l \quad (51)$$

and

$$\left(\frac{\partial\delta_2}{\partial l}\right)_{P_2} = \left(\frac{\partial G}{\partial P_2}\right)_l \quad (52)$$

Hence, with $P_1 = P_2 = 0$, these lead to

$$\frac{\partial\delta_2(\hat{x}_1)}{\partial l} + i\frac{\partial\delta_1(\hat{x}_1)}{\partial l} = \frac{(c_1 + c_2)(F_2 + iF_1)(\cos\lambda - i\sin\lambda)}{4\pi(l - \tilde{x}_1)^{\frac{1}{2}}(l - \hat{x}_1)^{\frac{1}{2}}} \quad (53)$$

where

$$\zeta = \ln\left(\frac{l - \tilde{x}_1}{l - \hat{x}_1}\right)^\varepsilon \quad (54)$$

We convert Eq. (53) to the result for the cohesive zone with uniform stress. This leads to

$$\frac{\partial\delta_2(\hat{x}_1)}{\partial l} + i\frac{\partial\delta_1(\hat{x}_1)}{\partial l} = -\frac{(c_1 + c_2)(\sigma_c + i\sigma_s)}{4\pi(l - \hat{x}_1)^{\frac{1}{2}}} \int_0^l \frac{(\cos\zeta - i\sin\zeta) d\tilde{x}_1}{(l - \tilde{x}_1)^{\frac{1}{2}}} \quad (55)$$

With ε small,

$$\cos\zeta \simeq 1 - \frac{\zeta^2}{2} = 1 - \varepsilon^2 \ln\left(\frac{l - \tilde{x}_1}{l - \hat{x}_1}\right) \quad (56)$$

and $\sin\zeta \simeq \varepsilon \ln\left(\frac{l - \tilde{x}_1}{l - \hat{x}_1}\right)$. Using this, we deduce that the numerator in the integrand will only have a small imaginary part and the real part will be close to unity, namely

$$\int_0^l \frac{(\cos\zeta) d\tilde{x}_1}{(l - \tilde{x}_1)^{\frac{1}{2}}} \simeq \int_0^l \frac{d\tilde{x}_1}{(l - \tilde{x}_1)^{\frac{1}{2}}} - \varepsilon^2 \int_0^l \frac{\ln\left(\frac{l - \tilde{x}_1}{l - \hat{x}_1}\right) d\tilde{x}_1}{(l - \tilde{x}_1)^{\frac{1}{2}}}$$

Hence,

$$\delta_2(\hat{x}_1) = -\frac{(c_1 + c_2)\sigma_c}{2\pi} \int_{\hat{x}_1}^b \frac{\hat{l}^{\frac{1}{2}} \left[1 + 2\varepsilon^2 \left(-2 + \ln\left(\frac{\hat{l}}{\hat{l} - \hat{x}_1}\right)\right)\right]}{(\hat{l} - \hat{x}_1)^{\frac{1}{2}}} d\hat{l} \quad (57)$$

With ε small, we retain only the first term not depending on ε^2 , and integrate to obtain

$$\delta_2(\hat{x}_1) \simeq -\frac{(c_1 + c_2)\sigma_c b}{2\pi} \left[\sqrt{\frac{b - \hat{x}_1}{b}} + \frac{\hat{x}_1}{2b} \ln \frac{1 + \sqrt{\frac{b - \hat{x}_1}{b}}}{1 - \sqrt{\frac{b - \hat{x}_1}{b}}} \right] \quad (58)$$

Now introduce $x = b - \hat{x}_1$ as the magnitude of the distance from the tip of the cohesive zone. The result above then becomes

$$\delta_2(x) \simeq -\frac{(c_1 + c_2)\sigma_c b}{2\pi} \left[\sqrt{\frac{x}{b}} + \frac{1}{2} \left(1 - \frac{x}{b}\right) \ln \frac{1 + \sqrt{\frac{x}{b}}}{1 - \sqrt{\frac{x}{b}}} \right] \quad (59)$$

Based on the same deductions, the applied stress intensity factor causes a cohesive zone stretch given approximately by

$$\delta_A(x) = (c_1 + c_2) K_A \sqrt{\frac{x}{8\pi}} = \frac{(c_1 + c_2) \sigma_c b}{\pi} \sqrt{\frac{x}{b}} \quad (60)$$

where eqt.39 has been used to eliminate K_A in favour of b . The total cohesive zone stretch is given by Eq. (59) added to Eq. (60), leading to

$$\delta_{2,tot}(x) \simeq \frac{(c_1 + c_2) \sigma_c b}{2\pi} \left[\sqrt{\frac{x}{b}} - \frac{1}{2} \left(1 - \frac{x}{b}\right) \ln \frac{1 + \sqrt{\frac{x}{b}}}{1 - \sqrt{\frac{x}{b}}} \right] \quad (61)$$

as assumed for our calculations (8). In retaining the second order term,

$$\delta_{2,tot}^{2nd}(x) \simeq \frac{(c_1 + c_2) \sigma_c b}{2\pi} \left[\sqrt{\frac{x}{b}} - \frac{1}{2} \left(1 - \frac{x}{b}\right) \ln \frac{1 + \sqrt{\frac{x}{b}}}{1 - \sqrt{\frac{x}{b}}} - \frac{\varepsilon^2}{b} \int_{\hat{x}_1}^b \frac{2\hat{l}^{\frac{1}{2}} \left(-2 + \ln \left(\frac{\hat{l}}{\hat{l} - \hat{x}_1}\right)\right)}{\left(\hat{l} - \hat{x}_1\right)^{\frac{1}{2}}} d\hat{l} \right] \quad (62)$$

Note that c_1 gives the contribution due to the upper material and c_2 that of the lower material.

Fig.4 shows the cohesive zone stretch due to the cohesive zone tractions only without the cohesive zone stretch due to the applied stress intensity factor i.e. from Eq.(57) made dimensionless as $\widehat{\delta}_2(x) \simeq \delta_2(x) / \left(-\frac{(c_1+c_2)\sigma_c b}{2\pi}\right)$ for $\varepsilon = 0$, (black thick solid line), and for $\varepsilon = 0.1, 0.2, 0.3$ to second order with blue, red and green lines. While the slope of the black line should be vertical at x/b the effect is very confined so that it is not apparent on the scale of Fig. 4. Notice that it can be shown by some lengthly algebra the stretch of the crack outside the cohesive zone can be found by considering absolute values of the quantities in the log terms, so we are plotting this result too in Fig.4.

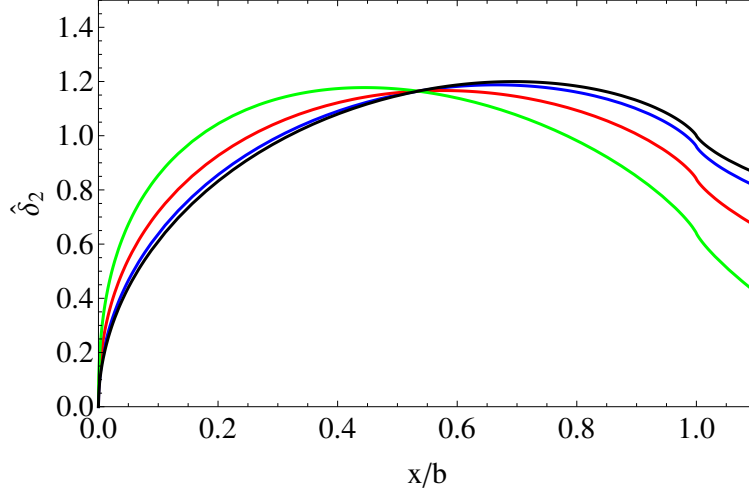


Figure 4 - Stretch due to the cohesive zone tractions only from Eq.(57) for $\varepsilon = 0$, (black thick solid line), and for $\varepsilon = 0.1, 0.2, 0.3$ to second order with blue, red and green lines.

10. Appendix 2 - Derivation of cohesive zone stretch result with viscoelasticity

The viscoelastic constitutive law is given by

$$\varepsilon_{ij}(t) = \int_{-\infty}^t C_{ijkl}(t-\tau) \frac{d\sigma_{kl}(\tau)}{dt} d\tau \quad (63)$$

where ε_{ij} is the strain, t is time, $C_{ijkl}(t)$ is the viscoelastic compliance and σ_{kl} is the stress. Within a homogeneous material, Eq. (63) may be rewritten as

$$\varepsilon_{ij}(t) = \int_{-\infty}^t C_{ijkl}(t-\tau) C_{klmn}^{-1}(0) \frac{d\varepsilon_{mn}^{el}(\tau)}{dt} d\tau \quad (64)$$

where $C_{klmn}^{-1}(0)$ is the tensor of instantaneous elastic moduli and $\varepsilon_{mn}^{el}(\tau)$ is the elastic strain at the current stress. We can integrate this with respect to position in the homogeneous material to obtain

$$u_i(t) = \int_{-\infty}^t \frac{C(t-\tau)}{C(0)} \frac{du_i^{el}(\tau)}{dt} d\tau \quad (65)$$

where $C(t-\tau)$ is an appropriate measure of the viscoelastic compliance and $C(0)$ is the equivalent elastic compliance.

Following Rice's (1978) treatment of the homogeneous case, we apply Eq. (65) to each side of a crack along a bimaterial interface. We consider incompressible materials, so that, after approximation of the cohesive zone stretch, the contribution to the elastic stretch due to material 1 which is in the upper half of the plane, in a small cohesive zone on a semi-infinite crack, is

$$\delta_1^{el} = \frac{\sigma_c b}{\pi G_1} \left[\sqrt{\frac{x}{b}} - \frac{1}{2} \left(1 - \frac{x}{b} \right) \ln \frac{1 + \sqrt{\frac{x}{b}}}{1 - \sqrt{\frac{x}{b}}} \right] = \frac{\sigma_c b}{\pi G_1} f(\lambda) \quad (66)$$

where G_1 is the shear modulus of material 1, $\lambda = x/b$, with an equivalent result for δ_2 , the contribution from material 2 in the lower half plane. In Eq.66, x is measured from the tip of the cohesive zone where $\delta_1^{el} = 0$ and $x = b$ at the tip of the crack where the cohesive zone will break. Thus, the length of the cohesive zone is b and we have, as an approximation,

$$b = \frac{\pi}{8} \left(\frac{K_A}{\sigma_c} \right)^2 \quad (67)$$

where K_A is the applied Mode I stress intensity factor, used as an approximation of the complex stress intensity factor for a bimaterial crack, and σ_c is the cohesive stress.

We define the viscoelastic compliance by

$$\gamma(t) = \int_{-\infty}^t C(t-\tau) \frac{d\sigma_s(\tau)}{d\tau} d\tau \quad (68)$$

where γ is shear strain and σ_s is shear stress. Therefore, $C_1(0) = 1/G_1$ and thus the viscoelastic contribution of material 1 to the cohesive zone stretch is

$$\delta_1(t) = G_1 \int_{-\infty}^t C_1(t-\tau) \frac{d\delta_1^{el}(\tau)}{d\tau} d\tau \quad (69)$$

Now consider a crack growing on the bimaterial interface, taken to be the x_1 -axis. The crack is growing at a rate V such that its tip, where the cohesive zone breaks, is at $x_1 = Vt$ and the tip of the cohesive zone is at $x_1 = Vt + b$. As a result,

$$x = Vt + b - x_1 \quad (70)$$

We consider steady state growth, with K_A and b both constant, so that

$$\frac{d\delta_1^{el}(x_1, t)}{dt} = \frac{\sigma_c b}{\pi G_1} \frac{\partial}{\partial t} f \left(\frac{Vt + b - x_1}{b} \right) \quad (71)$$

As a result, Eq. (69) becomes

$$\delta_1(x_1, t) = \frac{\sigma_c b}{\pi} \int_{-\infty}^t C_1(t - \tau) \frac{\partial}{\partial \tau} f\left(\frac{V\tau + b - x_1}{b}\right) d\tau \quad (72)$$

To obtain the crack tip stretch due to material 1 we set $x_1 = Vt$ so that

$$\delta_1(Vt, t) = \frac{\sigma_c b}{\pi} \int_{t - \frac{b}{V}}^t C_1(t - \tau) \frac{\partial}{\partial \tau} f\left(\frac{b - V(t - \tau)}{b}\right) d\tau \quad (73)$$

where the lower limit on the integration arises as that is the time material point that is the crack tip at time t entered the cohesive zone. We note that

$$\lambda = \frac{b - V(t - \tau)}{b} \quad (74)$$

Thus, we can write

$$\tau - t = \frac{b}{V}(\lambda - 1) \quad (75)$$

enabling a change of the integration variable so that Eq. (73) becomes

$$\delta_1^{tip} = \frac{\sigma_c b}{\pi} \int_0^1 C_1\left(\frac{b}{V}(\lambda - 1)\right) \frac{df(\lambda)}{d\lambda} d\lambda \quad (76)$$

where, as above,

$$f(\lambda) = \sqrt{\lambda} - \frac{1}{2}(1 - \lambda) \ln \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \quad (77)$$

and thus

$$\frac{df(\lambda)}{d\lambda} = \frac{1}{2} \ln \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \quad (78)$$

The equivalent result for material 2 is

$$\delta_2^{tip} = \frac{\sigma_c b}{\pi} \int_0^1 C_2\left(\frac{b}{V}(\lambda - 1)\right) \frac{df(\lambda)}{d\lambda} d\lambda \quad (79)$$

Note that for a Maxwell material

$$C(t) = \frac{1}{G} + \frac{t}{\mu} \quad (80)$$

where μ is the viscosity. Consequently,

$$\int_0^1 C_1\left(\frac{b}{V}(\lambda - 1)\right) \frac{df(\lambda)}{d\lambda} d\lambda = \frac{1}{G_1} + \frac{b}{3\mu_1 V} \quad (81)$$

It follows that the crack tip stretch for a Maxwell bimaterial crack where the materials are incompressible is

$$\delta^{tip} = \delta_1^{tip} + \delta_2^{tip} = \left[\frac{1}{G_1} + \frac{1}{G_2} + \frac{b}{3V} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] \frac{\sigma_c b}{\pi} \quad (82)$$